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PATHWAY FRACTIONAL INTEGRATION OPERATOR

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Abstract

In this paper we introduce a new fractional integration operator associated with the pathway model and pathway probability density of A.M. Mathai. This operator generalizes the classical Riemann-Liouville fractional integration operator, and also it can be reduced to the Laplace integral transform when the pathway parameter $\alpha \to 1$. We propose some results concerning images of special functions under the pathway operator.

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1. Introduction

Several definitions of the operators of the classical and generalized fractional calculus (FC) are already well known and widely used in the applications to mathematical models of fractional order. The most popular one, we are based on here, is the so-called Riemann-Liouville fractional integral (see for example the encyclopedia of FC by Samko-Kilbas-Marichev [18]). As operators of classical FC, also the Erdélyi-Kober operators are well known (cf. [18], Kiryakova [10]; Kober [9], Erdélyi [2]). Recently the so-called operators of the generalized FC have been introduced and studied, as the hypergeometric integral operators of Saigo [17], the generalized fractional integrals involving G- and H-functions in Kiryakova [10], etc. Here we introduce a fractional integration operator, which may be regarded as an extension of the left-sided Riemann-Liouville fractional integral operator. We propose some results for this operator, including the images of the Fox H-function, Mittag-Leffler function, Bessel function of first kind, and their particular cases. Also, the Mellin transform of the pathway integral operator is considered.

One of the important aspects of the results obtained in this paper is that a connection is established to wide classes of statistical distributions. The pathway parameter α establishes a path of going from one distribution to another and to different classes of distributions. Thus, the new operator introduced in this paper, will enable us to derive a number of results covering wide range of distributions. The "fractional integration" nature of the operator will then extend the corresponding results to wider ranges, where when the pathway parameter α goes to 1 the corresponding results on generalized gamma type functions are obtained.

First, let us recall the definition of left sided Riemann-Liouville fractional integral operator. Let $f(x) \in L(a,b),\ \alpha \in \mathbb{C},\ \Re(\alpha) > 0$, then

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \tag{1}$$

where $\Re(\cdot)$ denotes the real part of $(\cdot)$. For more details, we refer to: Samko-Kilbas-Marichev [18], Kiryakova [10], Kilbas-Srivastava-Trujillo [8], and other books on FC.

If $f(t)$ is replaced by $t^{\gamma} f(t)$ in (1), the above operator turns out to be the Erdélyi-Kober fractional integral; if it is replaced by ${}_2F_1(\eta+\beta,\ -\gamma;\eta;1-\frac{t}{x})\ f(t)$, then (1) takes the form of the Saigo hypergeometric fractional integral, see e.g. [17]:

$$\frac{\Gamma(\eta)}{x^{-\eta-\beta}}\, I_{0+}^{\eta,\beta,\gamma} f(x) = \int_0^x (x-t)^{\eta-1}\ {}_2F_1(\eta+\beta,-\gamma;\eta;1-\frac{t}{x}) f(t)\mathrm{d}t.$$

Many other operators of generalized fractional calculus can be obtained if on the place of $f(t)$ one takes $\Phi(t)f(t)$ with a suitably chosen special function $\Phi(t)$, as it is done in Kiryakova [10] for a Fox's H-function $\Phi(t) = H_{m,m}^{m,0}(t)$.

In this paper we introduce an integral transform that can be considered as a new fractional integration operator.

DEFINITION 1.1. Let $f(x) \in L(a,b)$, $\eta \in \mathbb{C}$, $\Re(\eta) > 0$, $a > 0$ and let us take a "pathway parameter" $\alpha < 1$. Then the *pathway fractional integration operator*, as an extension of (1), is defined as follows:

$$(P_{0_+}^{(\eta,\alpha)} f)(x) = x^\eta \int_0^{[\frac{x}{a(1-\alpha)}]} [1 - \frac{a(1-\alpha)t}{x}]^{\frac{\eta}{(1-\alpha)}} f(t) \mathrm{d}t. \tag{2}$$

The *pathway model* is introduced by Mathai [13] and studied further by Mathai and Haubold [14],[15]. For real scalar α, the pathway model for scalar random variables is represented by the following *probability density function* (p.d.f.):

$$f(x) = c\ |x|^{\gamma-1}\ [1 - a(1-\alpha)|x|^\delta]^{\frac{\beta}{1-\alpha}}, \tag{3}$$

$-\infty < x < \infty$, $\delta > 0$, $\beta \geq 0$, $1 - a(1-\alpha)|x|^\delta > 0$, $\gamma > 0$, where c is the normalizing constant and α is called the *pathway parameter*. For real α, the normalizing constant is as follows:

$$c = \frac{1}{2} \frac{\delta [a(1-\alpha)]^{\frac{\gamma}{\delta}}\ \Gamma(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1)}{\Gamma(\frac{\gamma}{\delta})\ \Gamma(\frac{\beta}{1-\alpha} + 1)}, \text{ for } \alpha < 1, \tag{4}$$

$$= \frac{1}{2} \frac{\delta [a(\alpha-1)]^{\frac{\gamma}{\delta}}\ \Gamma(\frac{\beta}{\alpha-1})}{\Gamma(\frac{\gamma}{\delta})\ \Gamma(\frac{\beta}{\alpha-1} - \frac{\gamma}{\delta})}, \text{ for } \frac{1}{\alpha-1} - \frac{\gamma}{\delta} > 0,\ \alpha > 1, \tag{5}$$

$$= \frac{1}{2} \frac{\delta (a\ \beta)^{\frac{\gamma}{\delta}}}{\Gamma(\frac{\gamma}{\delta})} \text{ for } \alpha \to 1. \tag{6}$$

Observe that for $\alpha < 1$ it is a finite range density with $1-a(1-\alpha)|x|^\delta > 0$ and (3) remains in the extended generalized type-1 beta family. The pathway density in (3), for $\alpha < 1$, includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d.f.

For $\alpha > 1$, writing $1 - \alpha = -(\alpha - 1)$ we have

$$f(x) = c\ |x|^{\gamma-1}\ [1 + a(\alpha-1)|x|^\delta]^{-\frac{\beta}{\alpha-1}}, \tag{7}$$

$-\infty < x < \infty$, $\delta > 0$, $\beta \geq 0$, $\alpha > 1$ which is the extended generalized type-2 beta model for real x. It includes the type-2 beta density, the F density, the Student-t density, the Cauchy density and many more.

Here we consider *only the case of pathway parameter* $\alpha < 1$. For $\alpha \to 1$ both (3) and (7) take the exponential form, since

$$\begin{aligned}\lim_{\alpha\to 1} c|x|^{\gamma-1}[1-a(1-\alpha)|x|^{\delta}]^{\frac{\eta}{1-\alpha}} &= \lim_{\alpha\to 1} c|x|^{\gamma-1}[1+a(\alpha-1)|x|^{\delta}]^{-\frac{\eta}{\alpha-1}} \\ &= c|x|^{\gamma-1}\mathrm{e}^{-a\eta|x|^{\delta}}. \end{aligned} \tag{8}$$

This includes the generalized gamma-, the Weibull-, the chi-square, the Laplace-, Maxwell-Boltzmann and other related densities. Therefore, the operator introduced in this paper can be related and applicable to a wide variety of statistical densities.

For more details on the pathway model, the reader is referred to the recent papers of Mathai and Haubold [14],[15].

When $\alpha \to 1_-$, $[1-\frac{a(1-\alpha)t}{x}]^{\frac{\eta}{1-\alpha}} \to \mathrm{e}^{-\frac{a\eta}{x}t}$. Then, operator (2) gets the form

$$(P_{0_+}^{(\eta,1)}f)(x) = x^{\eta}\int_0^{\infty} \mathrm{e}^{-\frac{a\eta}{x}t}\; f(t)\;\mathrm{d}t = x^{\eta}\; L_f(\frac{a\eta}{x}), \tag{9}$$

that is, *it reduces to the Laplace integral transform* of f with parameter $\frac{a\eta}{x}$:

$$L_f(x) = \int_0^{\infty} e^{-xt}\, f(t)dt. \tag{10}$$

When $\alpha = 0,\ a = 1$, then replacing η by $\eta - 1$ in (2) the integral operator gets the form

$$\int_0^x (x-t)^{\eta-1} f(t)\mathrm{d}t = \Gamma(\eta)(I_{0_+}^{\eta} f)(x), \tag{11}$$

and *reduces to the left-sided Riemann-Liouville fractional integral* $I_{0_+}^{\eta}$ in (1).

It is seen that the pathway fractional integral operator (2), based on the pathway model of Mathai and Haubold, and using the pathway parameter α, can lead to other interesting examples of FC operators, related to some probability density functions and applications in statistics.

Lemma 1. *Let* $\eta \in \mathbb{C}$, $\Re(\eta) > 0$, $\beta \in \mathbb{C}$ *and* $\alpha < 1$. *If* $\Re(\beta) > 0$, $\Re(\frac{\eta}{1-\alpha}) > -1$, *then there holds the basic formula*

$$P_{0_+}^{(\eta,\alpha)}[x^{\beta-1}] = \frac{x^{\eta+\beta}}{[a(1-\alpha)]^{\beta}}\,\frac{\Gamma(\beta)\;\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\frac{\eta}{1-\alpha}+\beta+1)}\,. \tag{12}$$

P r o o f.

$$P_{0_+}^{(\eta,\alpha)}[x^{\beta-1}] = x^{\eta}\int_0^{\frac{x}{a(1-\alpha)}}[1-\frac{a(1-\alpha)t}{x}]^{\frac{\eta}{1-\alpha}}t^{\beta-1}\mathrm{d}t$$

$$= \frac{x^{\eta+\beta}}{[a(1-\alpha)]^{\beta}}\frac{\Gamma(\beta)\,\Gamma(1+\frac{\eta}{1-\alpha})}{\Gamma(\frac{\eta}{1-\alpha}+\beta+1)}, \quad \alpha<1,\ \Re(\eta)>0,\ \Re(\beta)>0.$$

■

In (12) when $\alpha \to 1_-$, $\frac{\eta}{1-\alpha} \to \infty$ and hence we may expand the gamma-functions by using the Stirling formula:

$$\frac{\Gamma(\frac{\eta}{1-\alpha}+1)}{(1-\alpha)^{\beta}\,\Gamma(\frac{\eta}{1-\alpha}+\beta+1)} \to \frac{\sqrt{2\pi}\,(\frac{\eta}{1-\alpha})^{\frac{\eta}{1-\alpha}+1-\frac{1}{2}}\,\mathrm{e}^{-\frac{\eta}{1-\alpha}}}{(1-\alpha)^{\beta}\,\sqrt{2\pi}\,(\frac{\eta}{1-\alpha})^{\frac{\eta}{1-\alpha}+\beta+1-\frac{1}{2}}\,\mathrm{e}^{-\frac{\eta}{1-\alpha}}} = \frac{1}{\eta^{\beta}}.$$

Hence

$$\lim_{\alpha\to 1_-} P_{0_+}^{(\eta,\alpha)}[x^{\beta-1}] \to \frac{\Gamma(\beta)\,x^{\eta+\beta}}{(a\eta)^{\beta}}. \tag{13}$$

This is the same as the Laplace transform formula, given by

$$\{\lim_{\alpha\to 1_-} P_{0_+}^{(\eta,\alpha)}\}[x^{\beta-1}] = x^{\eta}\int_0^{\infty}\mathrm{e}^{-\frac{a\eta}{x}t}\,t^{\beta-1}\mathrm{d}t = \frac{\Gamma(\beta)\,x^{\eta+\beta}}{(a\eta)^{\beta}},$$

for $x, a, \eta > 0, \Re(\beta) > 0$.

2. Pathway integral operator of an H-function

Charles Fox [5] made a detailed study of a Mellin-Barnes integral, which is now known in the literature as *Fox's H-function.* This function is defined and represented by means of the following Mellin-Barnes type contour integral

$$H_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\ \\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right] = \frac{1}{2\pi i}\int_L \phi(s)\ z^{-s}\mathrm{d}s, \quad i=+\sqrt{-1}, \tag{14}$$

where

$$\phi(s) = \frac{\{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\}\ \{\prod_{j=1}^{n}\Gamma(1-a_j-\alpha_j s)\}}{\{\prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)\}\ \{\prod_{j=n+1}^{p}\Gamma(a_j+\alpha_j s)\}}$$

with $\alpha_j,\ j=1,2,...,p,\ \ \beta_j,\ j=1,2,...,q$ – real positive numbers, $a_j,\ j=1,2,...,p$ and $b_j,\ j=1,2,...,q$ – complex numbers, L – a contour separating the poles of $\Gamma(b_j+\beta_j s),\ j=1,2,...,m$ from those of $\Gamma(1-a_j-\alpha_j s),\ j=1,2,...,n$. For the convergence conditions, existence of various contours L and other properties, see Mathai and Saxena [16], Kilbas and Saigo [6], Kilbas-Srivastava-Trujillo [8], etc. The importance of Fox's H-function lies in the fact that almost all the elementary and special functions in the literature follow as its special cases. These special functions appear in various problems arising in theoretical and applied branches of mathematics, statistics, physics, engineering and other areas.

THEOREM 1. *Let $\eta,\rho\in\mathbb{C},\Re(\beta)>0,\ \Re(1+\frac{\eta}{1-\alpha})>0,\ \Re(\rho)>0$ and $\alpha<1,\ b\in\mathbb{R}$. Then for the pathway fractional integral $P_{0_+}^{(\eta,\alpha)}$ the following formula holds for the image of an arbitrary Fox H-function:*

$$\left(P_{0_+}^{(\eta,\alpha)}t^{\rho-1}H_{p,q}^{m,n}\left[bt^{\beta}\Big|_{(b_1,\beta_1),...,(b_q,\beta_q)}^{(a_1,\alpha_1),...,(a_p,\alpha_p)}\right]\right)=\frac{x^{\eta+\rho}\,\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}}$$
$$\times H_{p+1,q+1}^{m,n+1}\left[\frac{bx^{\beta}}{(a(1-\alpha))^{\beta}}\Big|_{(b_1,\beta_1),...,(b_q,\beta_q),(-\rho-\frac{\eta}{1-\alpha},\beta)}^{(a_1,\alpha_1),...,(a_p,\alpha_p),(1-\rho,\beta)}\right]. \tag{15}$$

P r o o f. Using the definitions (2) and (14), we have

$$\left(P_{0_+}^{(\eta,\alpha)}t^{\rho-1}H_{p,q}^{m,n}\left[bt^{\beta}\Big|_{(b_1,\beta_1),...,(b_q,\beta_q)}^{(a_1,\alpha_1),...,(a_p,\alpha_p)}\right]\right)$$
$$=x^{\eta}\int_0^{\frac{x}{a(1-\alpha)}}t^{\rho-1}[1-\frac{a(1-\alpha)t}{x}]^{\frac{\eta}{1-\alpha}}\frac{1}{2\pi i}\int_L\phi(s)(bt^{\beta})^{-s}\mathrm{d}s\mathrm{d}t.$$

Interchanging the order of integration and evaluating the integral by the beta function formula, it gives

$$=\frac{x^{\eta+\rho}\,\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}}\,\frac{1}{2\pi i}\int_L\phi(s)\frac{\Gamma(\rho-\beta s)}{\Gamma(1+\rho+\frac{\eta}{1-\alpha}-\beta s)}\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right]^{-s}\mathrm{d}s$$
$$=\frac{x^{\eta+\rho}\,\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}}\,H_{p+1,q+1}^{m,n+1}\left[\frac{bx^{\beta}}{(a(1-\alpha))^{\beta}}\Big|_{(b_1,\beta_1),...,(b_q,\beta_q),(-\rho-\frac{\eta}{1-\alpha},\beta)}^{(a_1,\alpha_1),...,(a_p,\alpha_p),(1-\rho,\beta)}\right].$$

The interchange of the order of integration is permissible under the conditions stated in the theorem due to convergence of the integrals involved in the process. This completes the proof of Theorem 1. ∎

When $\alpha \to 1_-$, then (15) tends to

$$\frac{x^{\eta+\rho}\,\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}}\,\frac{1}{2\pi i}\int_L \phi(s)\frac{\Gamma(\rho-\beta s)}{\Gamma(1+\rho+\frac{\eta}{1-\alpha}-\beta s)}\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right]^{-s}\mathrm{d}s \to$$

$$\to \frac{x^{\eta+\rho}}{(a\eta)^{\rho}}\frac{1}{2\pi i}\int_L \Phi(s)\Gamma(\rho-\beta s)\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right]^{-s}\mathrm{d}s$$

$$\to \frac{x^{\eta+\rho}}{(a\eta)^{\rho}}H^{m,n+1}_{p+1,q}\left[\frac{bx^{\beta}}{a^{\beta}\eta^{\beta}}\middle|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p),(1-\rho,\beta)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right], \tag{16}$$

as it gives also the formula for Laplace transform of a H-function.

3. Pathway integral operator of Mittag-Leffler and Bessel functions and their special cases

The special function of the form

$$E_{\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+1)},\quad \beta\in\mathbb{C},\ \Re(\beta)>0, \tag{17}$$

and the more general 2-indices function

$$E_{\beta,\rho}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta k+\rho)},\quad \beta,\ \rho\in\mathbb{C},\ \Re(\rho)>0,\ \Re(\beta)>0, \tag{18}$$

are known as the *Mittag-Leffler functions.* Recently, the interest to these functions has grown due to their applications in some reaction-diffusion problems and their various generalizations appearing in the solutions of fractional order differential and integral equations. The basic facts about the classical Mittag-Leffler (M-L) functions can be found yet in the handbook of Erdélyi et al. [4], and more deep results are given in the book by Dzherbashyan [1]. Some more recent results and generalizations of the M-L functions (17),(18) can be found in the works by Kiryakova [10] and [11], Kilbas-Saigo-Saxena [7], Kilbas-Srivastava-Trujillo [8] and many surveys and papers dealing with fractional order equations and systems and their applications.

One example of a generalized M-L function, which is a particular case of the Wright function, is the function studied by Kilbas-Saigo-Saxena [7] and Kilbas-Srivastava-Trujillo [8]:

$$E^{\gamma}_{\beta,\rho}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\beta k+\rho)}\frac{z^k}{k!},\quad \rho,\beta,\gamma\in\mathbb{C},\ \Re(\rho)>0,\ \Re(\beta)>0, \tag{19}$$

where $(\gamma)_k$ is the Pochhammer symbol

$$(\gamma)_k = \begin{cases} 1, k = 0 \\ \gamma(\gamma+1)\cdots(\gamma+k-1), \ k = 1, 2, \cdots, \ \gamma \neq 0. \end{cases} \tag{20}$$

For $\gamma = 1$, (19) coincides with (18), while for $\gamma = \beta = 1$ coincides with (17):

$$E^1_{\beta,\rho}(z) = E_{\beta,\rho}(z), \quad E^1_{\beta,1}(z) = E_\beta(z).$$

The Mittag-Leffler-type functions belong to H-functions family, since they can be represented in terms of the H-function:

$$E_\beta(z) = H^{1,1}_{1,2}\left[-z\Big|^{(0,1)}_{(0,1),(0,\beta)}\right], \ \beta \in \mathbb{C}, \ \Re(\beta) > 0, \tag{21}$$

$$E_{\beta,\rho}(z) = H^{1,1}_{1,2}\left[-z\Big|^{(0,1)}_{(0,1),(1-\rho,\beta)}\right], \ \beta, \rho \in \mathbb{C}, \ \Re(\beta) > 0, \tag{22}$$

$$E^\gamma_{\beta,\rho}(z) = \frac{1}{\Gamma(\gamma)} H^{1,1}_{1,2}\left[-z\Big|^{(1-\gamma,1)}_{(0,1),(1-\rho,\beta)}\right], \ \rho, \beta, \gamma \in \mathbb{C}, \ \Re(\beta) > 0. \tag{23}$$

In this section we establish the images under the pathway integral operator for the functions of M-L-type.

Theorem 2. *Let $\eta, \gamma, \rho \in \mathbb{C}$, $\Re(\eta) > 0, \Re(\gamma) > 0, \Re(\rho) > 0$, $\Re(1 + \frac{\eta}{1-\alpha}) > \max[0, -\Re(\rho)]$, $b \in \mathbb{R}$, $\beta > 0$ and $\alpha < 1$. Then the image of (19) under the pathway integral operator $P^{(\eta,\alpha)}_{0_+}$ is given by the formula:*

$$\left(P^{(\eta,\alpha)}_{0_+} x^{\rho-1} \ E^\gamma_{\beta,\rho}(bx^\beta)\right) = \frac{x^{\eta+\rho} \ \Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^\rho} E^\gamma_{\beta,1+\rho+\frac{\eta}{1-\alpha}}\left[\frac{bx^\beta}{(a(1-\alpha))^\beta}\right]. \tag{24}$$

P r o o f. This result can be derived from Theorem 1, by putting $m = 1$, $n = 1$, $p = 1$, $q = 2$, $b_1 = 0$, $\beta_1 = 1$, $b_2 = 1 - \rho$, $\beta_2 = \beta$, $a_1 = 1 - \gamma$, $\alpha_1 = 1$, then (15) reduces to

$$\left(P^{(\eta,\alpha)}_{0_+} x^{\rho-1} \ E^\gamma_{\beta,\rho}(bx^\beta)\right) = \frac{x^{\eta+\rho} \ \Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^\rho \ \Gamma(\gamma)} \times H^{1,2}_{2,3}\left[\frac{bx^\beta}{(a(1-\alpha))^\beta}\Big|^{(1-\gamma,1),(1-\rho,\beta)}_{(0,1),(1-\rho,\beta),(-\rho-\frac{\eta}{1-\alpha},\beta)}\right]. \tag{25}$$

The corresponding Mellin-Barnes integral representation is given by

$$= \frac{x^{\eta+\rho}\ \Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}\ \Gamma(\gamma)}\frac{1}{2\pi i}\int_L \frac{\Gamma(s)\ \Gamma(\gamma-s)}{\Gamma(1+\rho+\frac{\eta}{1-\alpha}-\beta s)}\ [\frac{-bx^{\beta}}{(a(1-\alpha))^{\beta}}]^{-s}\mathrm{d}s$$

$$= \frac{x^{\eta+\rho}\ \Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}} E^{\gamma}_{\beta,1+\rho+\frac{\eta}{1-\alpha}}[\frac{bx^{\beta}}{(a(1-\alpha))^{\beta}}].$$

■

In (24), when $\alpha \to 1_-$ we have

$$\lim_{\alpha\to 1_-} \frac{x^{\eta+\rho}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\beta k+\rho+1+\frac{\eta}{1-\alpha})\ k!}\left[\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}\right]^k$$

$$\to \frac{x^{\eta+\rho}\sqrt{2\pi}(\frac{\eta}{1-\alpha})^{\frac{\eta}{1-\alpha}+1-\frac{1}{2}}\mathrm{e}^{-\frac{\eta}{1-\alpha}}}{[a(1-\alpha)]^{\rho}} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\sqrt{2\pi}(\frac{\eta}{1-\alpha})^{\frac{\eta}{1-\alpha}+1+\beta k+\rho-\frac{1}{2}}\mathrm{e}^{-\frac{\eta}{1-\alpha}}} \times [\frac{bx^{\beta}}{[a(1-\alpha)]^{\beta}}]^k$$

$$= \frac{x^{\eta+\rho}}{(a\eta)^{\rho}} \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!}\left[\frac{bx^{\beta}}{(a\eta)^{\beta}}\right]^k = \frac{x^{\rho+\eta}(a\eta)^{\beta\gamma-\rho}}{(a^{\beta}\eta^{\beta}-bx^{\beta})^{\gamma}}, \qquad |\frac{bx^{\beta}}{(a\eta)^{\beta}}| < 1,$$

that is

$$\lim_{\alpha\to 1_-}\left(P^{(\eta,\alpha)}_{0_+} x^{\rho-1}\ E^{\gamma}_{\beta,\rho}(bx^{\beta})\right) = \frac{x^{\rho+\eta}(a\eta)^{\beta\gamma-\rho}}{(a^{\beta}\eta^{\beta}-bx^{\beta})^{\gamma}},$$

which *corresponds to the Laplace transform* of the function $x^{\rho-1}\ E^{\gamma}_{\beta,\rho}(bx^{\beta})$.

If $\gamma = 1$, (24) yields the *pathway fractional integral transform of the Mittag-Leffler function* (18):

$$\left(P^{(\eta,\alpha)}_{0_+} x^{\rho-1}\ E_{\beta,\rho}(bx^{\beta})\right) = \frac{x^{\eta+\rho}\ \Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\rho}} E_{\beta,1+\rho+\frac{\eta}{1-\alpha}}[\frac{bx^{\beta}}{(a(1-\alpha))^{\beta}}]. \tag{26}$$

For $\alpha \to 1_-$, (26) gives the Laplace transform image:

$$\lim_{\alpha\to 1_-}\left(P^{(\eta,\alpha)}_{0_+} x^{\rho-1}\ E_{\beta,\rho}(bx^{\beta})\right) = \frac{x^{\rho+\eta}(a\eta)^{\beta-\rho}}{a^{\beta}\eta^{\beta}-bx^{\beta}}.$$

For $\rho = 1$, (26) reduces to the pathway integral image of M-L function of form (17):

$$\left(P^{(\eta,\alpha)}_{0_+} E_{\beta}(bx^{\beta})\right) = \frac{x^{\eta+1}\ \Gamma(1+\frac{\eta}{1-\alpha})}{a(1-\alpha)} E_{\beta,1+\frac{\eta}{1-\alpha}}[\frac{bx^{\beta}}{(a(1-\alpha))^{\beta}}]. \tag{27}$$

When $\alpha \to 1_-$, it gives its Laplace image:

$$\lim_{\alpha\to 1_-}\left(P_{0_+}^{(\eta,\alpha)} E_{\beta,1}(bx^\beta)\right) = \frac{x^{1+\eta}(a\eta)^{\beta-1}}{a^\beta\eta^\beta - bx^\beta}.$$

Consider now the particular case when $\beta = 1$ and the M-L function (17) coincides with the exponential function: $E_{1,1}(bx) = \mathrm{e}^{bx}$. Then the following result holds.

Corollary 2.1. *Let $\eta \in \mathbb{C}$, $\Re(1+\frac{\eta}{1-\alpha}) > 0$, $\Re(\eta) > 0$, $b \in \mathbb{R}$ and $\alpha < 1$. Then the pathway fractional integral of the exponential function is given by the formula*

$$\left(P_{0_+}^{(\eta,\alpha)}\mathrm{e}^{bx}\right) = \frac{x^{\eta+1}\Gamma(1+\frac{\eta}{1-\alpha})}{a(1-\alpha)} E_{1,2+\frac{\eta}{1-\alpha}}[\frac{bx}{a(1-\alpha)}]. \tag{28}$$

If $\alpha \to 1_-$, then $\frac{\eta}{1-\alpha} \to \infty$ and by expanding the gamma-function by the Stirling approximation formula, (28) becomes

$$\frac{x^{\eta+1}\Gamma(1+\frac{\eta}{1-\alpha})}{a(1-\alpha)} E_{1,2+\frac{\eta}{1-\alpha}}[\frac{bx}{a(1-\alpha)}] \to \frac{x^{\eta+1}}{a\eta}\sum_{k=0}^{\infty}(\frac{bx}{a\eta})^k, \quad |\frac{bx}{a\eta}| < 1,$$

that is, as naturally expected, we obtain the Laplace image of the function e^{bt} with parameter $\frac{a\eta}{x}$:

$$\lim_{\alpha\to 1_-}\left(P_{0_+}^{(\eta,\alpha)}\mathrm{e}^{bx}\right) = \frac{x^{\eta+1}}{a\eta - bx} = \left(\lim_{\alpha\to 1} P_{0_+}^{(\eta,\alpha)}\right)\mathrm{e}^{bx}.$$

The *Bessel function* of the first kind $J_\nu(x)$ is defined for complex $x \in \mathbb{C}$, $x \neq 0$ and $\nu \in \mathbb{C}$, $\Re(\nu) > -1$ by

$$J_\nu(x) = (\frac{x}{2})^\nu \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{x}{2})^{2k}}{\Gamma(\nu+k+1)k!}, \tag{29}$$

and its H-function representation is given by

$$J_\nu(x) = (\frac{x}{2})^\nu H_{0,2}^{1,0}\left[\frac{x^2}{4}\Big|_{(0,1),(-\nu,1)}\right], \ \nu \in \mathbb{C},\ \Re(\nu) > 0.$$

The asymptotic behavior and other properties of this function can be seen from the papers of Wright [20],[21] and the handbook Erdélyi et al [3]. We find here its pathway integral image.

THEOREM 3. *Let* $\eta, \gamma, \nu \in \mathbb{C}$, $\Re(1+\frac{\eta}{1-\alpha}) > 0$, $\Re(\gamma+\nu) > 0$, $\Re(\eta) > 0$ *and* $\alpha < 1$. *Let* $P_{0_+}^{(\eta,\alpha)}$ *be the pathway fractional integral. Then there holds the image*

$$\left(P_{0_+}^{(\eta,\alpha)} (\frac{x}{2})^{\gamma-1} \; J_\nu(x)\right) = \frac{x^{\eta+\nu+\gamma} \; \Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\gamma+\nu} 2^{\nu+\gamma-1}} \times {}_1\Psi_2 \left[\left.{}^{(\gamma+\nu,2)}_{(\nu+1,1),(1+\gamma+\nu+\frac{\eta}{1-\alpha},2)}\right| -\frac{x^2}{4a^2(1-\alpha)^2}\right]. \quad (30)$$

P r o o f. The proof runs parallel to that in Theorem 2 above, but for the sake of completeness we give the outline here. This result can be obtained from Theorem 1 by putting $m=1$, $n=0$, $p=0$, $q=2$, $b_1=0$, $\beta_1=1$, $b_2=-\nu$, $\beta_2=1$, $\rho=\gamma+\nu$, $b=1$, $\beta=2$ and replacing x by $\frac{x}{2}$. Then (15) reduces to

$$\begin{aligned}\left(P_{0_+}^{(\eta,\alpha)} (\frac{x}{2})^{\gamma-1} \; J_\nu(x)\right) &= \frac{x^{\eta+\nu+\gamma} \; \Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\gamma+\nu} 2^{\nu+\gamma+\eta}} \\ &\quad \times H_{1,1}^{1,3}\left[\frac{x^2}{4a^2(1-\alpha)^2}\left|{}^{(1-\gamma-\nu,2)}_{(0,1),(-\nu,1),(-\nu-\gamma-\frac{\eta}{1-\alpha},2)}\right.\right] \\ &= \frac{x^{\eta+\nu+\gamma} \; \Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\gamma+\nu} 2^{\nu+\gamma+\eta}} \\ &\quad \times {}_1\Psi_2 \left[\left.{}^{(\gamma+\nu,2)}_{(\nu+1,1),(1+\gamma+\nu+\frac{\eta}{1-\alpha},2)}\right| -\frac{x^2}{4a^2(1-\alpha)^2}\right].\end{aligned}$$

Here ${}_p\Psi_q(x)$ denotes the *generalized Wright hypergeometric function* defined for $x \in \mathbb{C}$, $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}$ $(i=1,2,\dots,p;\ j=1,2,\dots,q)$ by the series

$${}_p\Psi_q(x) = {}_p\Psi_q\left[\left.{}^{(a_i,\alpha_i)_{1,p}}_{(b_j,\beta_j)_{1,q}}\right| x\right] \equiv \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j+\beta_j k)} \frac{x^k}{k!}. \quad (31)$$

■

When $\alpha \to 1_-$, (30) becomes

$$\lim_{\alpha\to 1_-} \left(P_{0_+}^{(\eta,\alpha)} (\frac{x}{2})^{\gamma-1} \; J_\nu(x)\right) = \frac{(\frac{x}{2})^{\eta+\nu+\gamma}}{(a\eta)^{\gamma+\nu}} \; {}_1\Psi_1\left[\left.{}^{(\gamma+\nu,2)}_{(\nu+1,1)}\right| -\frac{x^2}{4a^2\eta^2}\right].$$

Note that for $\nu = -\frac{1}{2}$, the Bessel function $J_\nu(x)$ in (29) coincides with cosine function, up to the multiplier $\left(\frac{2}{\pi z}\right)^{\frac{1}{2}}$ (see e.g. [3]):

$$J_{-\frac{1}{2}}(x) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos(x). \tag{32}$$

Corollary 3.1. *Let $\eta, \gamma \in \mathbb{C}$, $\Re(1+\frac{\eta}{1-\alpha}) > 0$, $\Re(\gamma) > 0$, $\Re(\eta) > 0$ and $\alpha < 1$. Let $P_{0_+}^{(\eta,\alpha)}$ be the pathway fractional integral. Then there holds the formula*

$$\begin{aligned}\left(P_{0_+}^{(\eta,\alpha)}(\frac{x}{2})^{\gamma-1}\cos x\right) &= \frac{\pi^{\frac{1}{2}} x^{\eta+\gamma}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\gamma}\ 2^{\gamma-1}} \\ &\times {}_1\Psi_2\left[\begin{matrix}(\gamma,2)\\(\frac{1}{2},1),(1+\gamma+\frac{\eta}{1-\alpha},2)\end{matrix}\middle|-\frac{x^2}{4a^2(1-\alpha)^2}\right].\end{aligned} \tag{33}$$

When $\alpha \to 1_-$, (33) reduces to

$$\lim_{\alpha\to 1_-}\left(P_{0_+}^{(\eta,\alpha)}(\frac{t}{2})^{\gamma-1}\ \cos t\right) \to \frac{\pi^{\frac{1}{2}} x^{\eta+\gamma}}{(a\eta)^{\gamma}\ 2^{\gamma-1}}\ {}_1\Psi_1\left[\begin{matrix}(\gamma,2)\\(\frac{1}{2},1)\end{matrix}\middle|-\frac{x^2}{4a^2\eta^2}\right].$$

Similarly, for $\nu = \frac{1}{2}$, the Bessel function $J_\nu(x)$ reduces to the sine function (see [3]):

$$J_{\frac{1}{2}}(x) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \sin(x). \tag{34}$$

Corollary 3.2. *Let $\eta, \gamma \in \mathbb{C}$, $\Re(1+\frac{\eta}{1-\alpha}) > 0$, $\Re(\gamma) > -1$, $\Re(\eta) > 0$ and $\alpha < 1$. Let $P_{0_+}^{(\eta,\alpha)}$ be the pathway fractional integral. Then there holds the formula*

$$\begin{aligned}\left(P_{0_+}^{(\eta,\alpha)}(\frac{x}{2})^{\gamma-1}\sin x\right) &= \frac{\pi^{\frac{1}{2}}}{2^{\gamma}}\frac{x^{\eta+\gamma+1}\Gamma(1+\frac{\eta}{1-\alpha})}{[a(1-\alpha)]^{\gamma+1}} \\ &\times {}_1\Psi_2\left[\begin{matrix}(\gamma+1,2)\\(\frac{3}{2},1),(2+\gamma+\frac{\eta}{1-\alpha},2)\end{matrix}\middle|-\frac{x^2}{4a^2(1-\alpha)^2}\right].\end{aligned} \tag{35}$$

For $\alpha \to 1_-$, (35) gives

$$\lim_{\alpha\to 1_-}\left(P_{0_+}^{(\eta,\alpha)}(\frac{x}{2})^{\gamma-1}\ \sin x\right) = \frac{\pi^{\frac{1}{2}} x^{\eta+\gamma+1}}{(a\eta)^{\gamma+1}\ 2^{\gamma}}\ {}_1\Psi_1\left[\begin{matrix}(\gamma+1,2)\\(\frac{3}{2},1)\end{matrix}\middle|-\frac{x^2}{4a^2\eta^2}\right].$$

REMARK 1. Note that the classical Mittag-Leffler functions (17),(18), the Bessel function (29) as well as its "fractional indices" extensions by Wright [20],[21] known as "Bessel-Maitland" or "Wright-Bessel" functions, are very special cases of the recently introduced *"multi-index Mittag-Leffler functions"* by Kiryakova [11], [12]:

$$E^{(m)}_{(\frac{1}{\rho_i}),(\mu_i)}(x)=\sum_{k=0}^{\infty}\frac{x^k}{\Gamma(\mu_1+k/\rho_1)\dots\Gamma(\mu_m+k/\rho_m)}. \tag{36}$$

They also belong to the family of H-functions, since:

$$E^{(m)}_{(\frac{1}{\rho_i}),(\mu_i)}(x)={}_1\Psi_m\left[\begin{array}{c}(1,1)\\(\mu_i,\frac{1}{\rho_i})_1^m\end{array}\middle| x\right]=H^{1,1}_{1,m+1}\left[-x\middle|\begin{array}{c}(0,1)\\(0,1),(1-\mu_i,\frac{1}{\rho_i})_1^m\end{array}\right].$$

Indeed, we have among their numerous particular cases (see Kiryakova [12]), the above-mentioned functions:

$$E_{1/\rho,\mu}(x)=E^{(1)}_{1/\rho,\mu}(x)\ ;\ J_\nu(x)=(x/2)^\nu E^{(2)}_{(1,1),(\nu+1,1)}(-x^2/4);$$
$$J^\mu_\nu(x)=E^{(2)}_{(1/\mu,1),(\nu+1,1)}(-x);\quad \text{etc.}$$

Therefore, in a way similar to the results provided here, and as a corollary of Theorem 1, one can find *the image of the multi-index M-L function (36) by the pathway integral operator.* Then, the images of many special functions (classical special functions and special functions of fractional calculus, see details in Kiryakova [12])) can be found as well.

4. Mellin transform of pathway operator

The following theorem establishes the relation between the *Mellin integral transform*:

$$M\{f(x);s\}=M\{f;s\}=\int_0^\infty x^{s-1}f(x)dx, \tag{37}$$

and the pathway integral operator.

THEOREM 4. *Let* $f(x)\in L_p(0,\infty)$, $\Re(\eta)>0$, $\Re(-\eta-s)>0$. *Then there holds the following result:*

$$M\{P^{(\eta,\alpha)}_{0_+}f;s\}=\frac{\Gamma(1+\frac{\eta}{1-\alpha})\ \Gamma(-\eta-s)\ [a(1-\alpha)]^{\eta+s}}{\Gamma(\frac{\eta\alpha}{1-\alpha}-s+1)}\,M\{f;1+s+\eta\}. \tag{38}$$

P r o o f.

$$M\{P_{0_+}^{(\eta,\alpha)}f;s\} = \int_0^\infty x^{s-1}\ x^{\eta} \int_0^{\frac{x}{a(1-\alpha)}} [1-\frac{a(1-\alpha)t}{x}]^{\frac{\eta}{1-\alpha}} f(t)\mathrm{d}t\ \mathrm{d}x$$

$$= [a(1-\alpha)]^{\eta+s} \int_0^\infty t^{\eta+s} f(t)\mathrm{d}t \int_0^\infty v^{\frac{\eta}{1-\alpha}} (1+v)^{-(\frac{\eta\alpha}{1-\alpha}-s+1)}\ \mathrm{d}v$$

$$= \frac{\Gamma(1+\frac{\eta}{1-\alpha})\ \Gamma(-\eta-s)\ [a(1-\alpha)]^{\eta+s}}{\Gamma(\frac{\eta\alpha}{1-\alpha}-s+1)}\ M\{f;1+s+\eta\},\quad \Re(-\eta-s)>0.$$

■

Remark 2. The new pathway fractional integration operator that we introduce and study here, is expected to have wide applications in statistical distribution theory. It could help to extend some classical statistical distributions to wider classes of distributions, useful in practical applications.

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References

[1] M.M. Dzherbashyan, *Integral Transforms and Representation of Functions in Complex Domain* (in Russian). Nauka, Moscow (1966).

[2] A. Erdélyi, On some functional transformation. *Univ. e. Politec. Torino, Rend. Sem. Mat.* **10**, (1940), 217-234.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions.* Vol. **II**. McGraw-Hill, New York (1953).

[4] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions.* Vol. **III**. McGraw-Hill, New York (1955).

[5] C. Fox, The G and H-functions as symmetrical Fourier kernels. *Trans. Amer. Math. Soc.* **98**, (1961), 395-429.

[6] A.A. Kilbas and M. Saigo, *H-Transforms: Theory and Applications*. Ser. 'Analytic Methods and Special Functions', Vol. **9**, CRC Press, New York (2004).

[7] A.A. Kilbas, M. Saigo and R.K. Saxena, Generalized Mittag-Leffler function and generalized fractional calculus operators. *Integral Transform. Spec. Funct.* **15**, (2004), 31-49.

[8] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam (2006).

[9] H. Kober, On fractional integrals and derivatives. *Quar. J. Math., Oxford Series* **II**, (1940), 193-211.

[10] V. Kiryakova, *Generalized Fractional Calculus and Applications*. Longman, Harlow & J. Wiley Sons, N. York (1994).

[11] V. Kiryakova, Multiindex Mittag-Leffler functions, related Gelfond-Leontiev operators and Laplace type transforms. *J. Comput. Appl. Mathematics* **118**, (2000), 241-259.

[12] V. Kiryakova, Some special functions related to fractional calculus and fractional (non-integer) order control systems and equations. *Facta Universitatis (Sci. J. of Univ.of Ni), Ser.: Automatic Control and Robotics*, **7**, No 1 (2008), 79-98 (UDC 517.93), available online at *http://facta.junis.ni.ac.rs/acar/acar200801/acar2008-07.pdf*

[13] A.M. Mathai, A pathway to matrix-variate gamma and normal densities. *Linear Algebra and Its Applications* **396**, (2005) 317-328.

[14] A.M. Mathai and H. J. Haubold, On generalized distributions and pathways. *Physics Letters* **372** (2008), 2109-2113.

[15] A.M. Mathai and H. J. Haubold, Pathway model, superstatistics, Tsallis statistics and a generalized measure of entropy. *Physica A* **375** (2007), 110-122.

[16] A.M. Mathai and R.K. Saxena, *The H-function with Applications in Statistics and Other Disciplines*. Wiley, New York (1978).

[17] M. Saigo, A remark on integral operators involving the Gauss hypergeometric functions. *Math. Rep. Kyushu Univ.* **11**, (1978), 135-143.

[18] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications.* Gordon and Breach, New York (1993).

[19] Seema S. Nair, Pathway fractional operator. *CMS Project SR/S4/MS: 287/05*, Preprint No. 48 (2008).

[20] E.M. Wright, The asymptotic expansion of the generalized Bessel function. *Proc. London Math. Soc.* **38**, (1934), 257-270.

[21] E.M. Wright, The generalized Bessel functions of order greater than one. *Quart. J. Math. Oxford Ser.* **11**, (1940), 36-48.

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